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2004 J. Phys. A: Math. Gen. 37 10983

(http://iopscience.iop.org/0305-4470/37/45/016)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 10983–10991 PII: S0305-4470(04)76942-5

Separation of variables for the nonlinear wave equation in polar coordinates

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Received 27 February 2004, in final form 28 September 2004
Published 28 October 2004
Online at stacks.iop.org/JPhysA/37/10983
doi:10.1088/0305-4470/37/45/016

Abstract
Some classical types of nonlinear wave motion in polar coordinates are studied within quadratic approximation. When the nonlinear quadratic terms in the wave equation are arbitrary, the usual perturbation techniques used in polar coordinates leads to overdetermined systems of linear algebraic equations for the unknown coefficients. However, we show that these overdetermined systems are compatible with the special case of the nonlinear shallow water equation and express explicitly the coefficients of the first two harmonics as polynomials of the Bessel functions of radius and of the trigonometric functions of angle. It gives a series of solutions to the nonlinear shallow water equation that are periodic in time and found with the same accuracy as the equation is derived.

PACS number: 02.30.Gp

1. Introduction

The two-dimensional nonlinear wave equation for a potential $\phi(x_1, x_2, t)$

$$-\kappa\phi_{tt} + \triangle\phi + \frac{\alpha}{2}(\nabla\phi\cdot\nabla\phi)_t + \frac{\beta}{2}\left(\phi_t^2\right)_t = 0 \tag{1}$$

describes the long surface water waves (see derivation in [3]) and the two-dimensional waves in an isentropic gas flow for the non-dissipative case (see [7]). Given in polar coordinates, equation (1) for $\varphi(r, \theta, t) = \phi(x_1, x_2, t)$ can be written as follows:

$$-\kappa\varphi_{tt} + \frac{1}{r}\varphi_r + \varphi_{rr} + \frac{1}{r^2}\varphi_{\theta\theta} + \frac{\alpha}{2}\left(\varphi_r^2 + \frac{1}{r^2}\varphi_\theta^2\right)_t + \frac{\beta}{2}\left(\varphi_t^2\right)_t = 0. \tag{2}$$

In his classic book *Hydrodynamics* [1, pp 191–5], Lamb considers at least three special cases of long *linear* waves in polar coordinates (θ, r). The first one is axisymmetric waves propagating over a horizontal bottom (3). The second one is the simplest unsymmetrical wave motion (4) in a circular basin. The third one gives a rough representation of the semidiurnal tide in a polar basin bounded by a small circle of latitude (5):

$$\varphi_1(r,\theta,t) = J_0(kr)\sin(\omega t), \tag{3}$$

$$\varphi_1(r,\theta,t) = J_1(kr)\cos\theta\sin(\omega t), \tag{4}$$

$$\varphi_1(r,\theta,t) = J_2(kr)\cos 2\theta\sin(\omega t), \tag{5}$$

where $k = \sqrt{\kappa\omega^2}$.

In this paper we present a family of approximate solutions to equation (2) in polar coordinates (θ, r) which gives, in particular, the next order *nonlinear* corrections to the well known linear solutions. These solutions are found with the same accuracy as the equation is derived. (Axisymmetric nonlinear waves were the subject of numerical investigations in a series of papers. See the bibliography in [4].) The specific corrections for linear solutions (3)–(5) and the corresponding numerical examples are given in section 4.

The potential $\varphi(r,\theta,t)$ is assumed to be regular in the vicinity of the origin of the coordinates and expanded in Fourier series with respect to t:

$$\varphi(r,\theta,t) = \varepsilon[S_1(r,\theta)\sin(\omega t) + C_1(r,\theta)\cos(\omega t)] + \varepsilon^2[S_2(r,\theta)\sin(2\omega t) + C_2(r,\theta)\cos(2\omega t)] + \cdots. \tag{6}$$

Then the functions $S_1(r,\theta)$ and $C_1(r,\theta)$ satisfy the Helmholz equation

$$Z_{rr} + \frac{1}{r}Z_r + \frac{1}{r^2}Z_{\theta\theta} + \kappa\omega^2 Z = 0, \tag{7}$$

and their Fourier expansions with respect to θ can be written as follows:

$$S_1(r,\theta) = a_0J_0(kr) + J_1(kr)(a_1\sin\theta + b_1\cos\theta) + \cdots + J_i(kr)(a_i\sin \mathrm{i}\theta + b_i\cos \mathrm{i}\theta) + \cdots, \tag{8}$$

$$C_1(r,\theta) = c_0J_0(kr) + J_1(kr)(c_1\sin\theta + d_1\cos\theta) + \cdots + J_i(kr)(c_i\sin \mathrm{i}\theta + d_i\cos \mathrm{i}\theta) + \cdots, \tag{9}$$

(see, for example [2]). In addition, we assume that series (8) and (9) are truncated and contain only N terms.

The functions $S_2(r,\theta)$ and $C_2(r,\theta)$ can be expanded in the following Fourier series:

$$S_2(r,\theta) = M_0(r) + \sum_{i=1}^{\infty}(M_i(r)\sin \mathrm{i}\theta + N_i(r)\cos \mathrm{i}\theta) \tag{10}$$

$$C_2(r,\theta) = P_0(r) + \sum_{i=1}^{\infty}(P_i(r)\sin \mathrm{i}\theta + Q_i(r)\cos \mathrm{i}\theta). \tag{11}$$

We search for the functions $M_i(r)$, $N_i(r)$, $P_i(r)$ and $Q_i(r)$ in the form

$$R_{00}J_0^2 + R_{01}J_0J_1 + R_{11}J_1^2, \tag{12}$$

where R_{AB} are polynomials of r^{-1} and r with unknown coefficients and of unknown degree n:

$$R_{AB} = \sum_{k=-n}^{n} C_k^{AB} r^k. \tag{13}$$

However, if a_i, b_i, c_i and d_i are fixed, substituting (10) and (11) in (2), we obtain *overdetermined* systems of linear algebraic equations for C_k^{AB}.

The key point of consideration is that these *overdetermined* systems are compatible. This allows us to construct explicit expressions for the functions S_2 and C_2 in the form (10) and (11), which are homogenous polynomials of the Bessel functions $J_0(kr)$, $J_1(kr)$ and the trigonometric functions of the angular variable θ. Their coefficients are polynomials of r^{-1} and r. The constructed series (6) and (8)–(11) can be considered as a generalization of the separation of variables method for the nonlinear wave equation.

2. Periodic problem, first reduction

Doing a suitable scaling, we can assume that $k = 1$ without loss of generality. Substituting (6) in (2), we obtain the following equations of the order ε^2:

$$S_{2rr} + \frac{1}{r}S_{2r} + \frac{1}{r^2}S_{2\theta\theta} + 4S_2 = \frac{\beta}{2\kappa\omega}\left(S_1^2 - C_1^2\right) - \frac{\alpha\omega}{2r^2}\left(S_{1\theta}^2 - C_{1\theta}^2\right) - \frac{\alpha\omega}{2}\left(S_{1r}^2 - C_{1r}^2\right), \qquad (14)$$

$$C_{2rr} + \frac{1}{r}C_{2r} + \frac{1}{r^2}C_{2\theta\theta} + 4C_2 = \frac{\beta}{\kappa\omega}S_1C_1 - \alpha\omega\left(\frac{S_{1\theta}C_{1\theta}}{r^2} + S_{1r}C_{1r}\right). \qquad (15)$$

Note that there is no terms independent of t in the order ε^2 since the nonlinear terms in (2) contain differentiation with respect to t. Our purpose is to find particular solutions to (14) and (15) in terms of Bessel functions. We rewrite these equations in the following form:

$$S_{2rr} + \frac{1}{r}S_{2r} + \frac{1}{r^2}S_{2\theta\theta} + 4S_2 = \lambda\left(S_1^2 - C_1^2\right) + \mu\left(\frac{1}{r^2}S_{1\theta}^2 - \frac{1}{r^2}C_{1\theta}^2 + S_{1r}^2 - C_{1r}^2\right), \qquad (16)$$

$$C_{2rr} + \frac{1}{r}C_{2r} + \frac{1}{r^2}C_{2\theta\theta} + 4C_2 = 2\lambda S_1C_1 + 2\mu\left(\frac{S_{1\theta}C_{1\theta}}{r^2} + S_{1r}C_{1r}\right). \qquad (17)$$

The right-hand sides of (16) and (17) can be represented as follows:

$$\lambda\left(S_1^2 - C_1^2\right) + \mu\left(\frac{1}{r^2}S_{1\theta}^2 - \frac{1}{r^2}C_{1\theta}^2 + S_{1r}^2 - C_{1r}^2\right) \equiv (\lambda - \mu)\left(S_1^2 - C_1^2\right)$$
$$+ \mu\left(\frac{1}{r^2}S_{1\theta}^2 - \frac{1}{r^2}C_{1\theta}^2 + S_{1r}^2 - C_{1r}^2 + S_1^2 - C_1^2\right)$$

$$2\lambda S_1C_1 + 2\mu\left(\frac{S_{1\theta}C_{1\theta}}{r^2} + S_{1r}C_{1r}\right) \equiv 2(\lambda - \mu)S_1C_1 + 2\mu\left(\frac{S_{1\theta}C_{1\theta}}{r^2} + S_{1r}C_{1r} + S_1C_1\right).$$

It can be easily checked that $\frac{1}{2}\left(S_1^2 - C_1^2\right)$ is a particular solution to the equation

$$S_{2rr} + \frac{1}{r}S_{2r} + \frac{1}{r^2}S_{2\theta\theta} + 4S_2 = S_1^2 - C_1^2 + \frac{1}{r^2}S_{1\theta}^2 - \frac{1}{r^2}C_{1\theta}^2 + S_{1r}^2 - C_{1r}^2 \qquad (18)$$

and $\frac{1}{2}S_1C_1$ is a particular solution to the equation

$$C_{2rr} + \frac{1}{r}C_{2r} + \frac{1}{r^2}C_{2\theta\theta} + 4C_2 = S_1C_1 + \frac{1}{r^2}S_{1\theta}C_{1\theta} + S_{1r}C_{1r}. \qquad (19)$$

So the problem is reduced to finding a particular solution to the equation

$$S_{2rr} + \frac{1}{r}S_{2r} + \frac{1}{r^2}S_{2\theta\theta} + 4S_2 = S_1^2 - C_1^2, \qquad (20)$$

and to the equation

$$C_{2rr} + \frac{1}{r}C_{2r} + \frac{1}{r^2}C_{2\theta\theta} + 4C_2 = S_1C_1, \qquad (21)$$

which is the purpose of the following section.

The general solution to (14) and (15) can be represented as a linear combination of the particular solution and the general solution to the homogenous equation

$$Z_{2rr} + \frac{1}{r} Z_{2r} + \frac{1}{r^2} Z_{2\theta\theta} + 4Z_2 = 0, \tag{22}$$

which can also be expressed in terms of the Bessel functions.

3. Reduction to ordinary differential equations

Substituting (8) and (9) in the right-hand side of (20) and (21), we have expressions of the following form:

$$\begin{aligned} S_1^2 - C_1^2 &= V_0^s(r) + \sum_{i=1}^{2N} \left(V_i^s(r) \sin \mathrm{i}\theta + W_i^s(r) \cos \mathrm{i}\theta \right) \\ S_1 C_1 &= V_0^c(r) + \sum_{i=1}^{2N} \left(V_i^c(r) \sin \mathrm{i}\theta + W_i^c(r) \cos \mathrm{i}\theta \right), \end{aligned} \tag{23}$$

where $V_i^s(r) W_i^s(r)$, $V_i^c(r)$ and $W_i^c(r)$ are known expressions of r. They are of the form

$$\begin{aligned} V_i^s(r) &= \sum_{j=0}^{i} \alpha_{ij}^{Vs} J_j(r) J_{i-j}(r) + \sum_{j=0}^{N-i} \beta_{ij}^{Vs} J_{j+i}(r) J_j(r), \\ W_i^s(r) &= \sum_{j=0}^{i} \alpha_{ij}^{Ws} J_j(r) J_{i-j}(r) + \sum_{j=0}^{N-i} \beta_{ij}^{Ws} J_{j+i}(r) J_j(r), \\ V_i^c(r) &= \sum_{j=0}^{i} \alpha_{ij}^{Vc} J_j(r) J_{i-j}(r) + \sum_{j=0}^{N-i} \beta_{ij}^{Vc} J_{j+i}(r) J_j(r), \\ W_i^s(r) &= \sum_{j=0}^{i} \alpha_{ij}^{Ws} J_j(r) J_{i-j}(r) + \sum_{j=0}^{N-i} \beta_{ij}^{Ws} J_{j+i}(r) J_j(r), \end{aligned} \tag{24}$$

where α_{ij}^{Vs}, β_{ij}^{Vs}, α_{ij}^{Ws}, β_{ij}^{Ws}, α_{ij}^{Vc}, β_{ij}^{Vc}, α_{ij}^{Wc} and β_{ij}^{Wc} are known constants.

We seek particular solutions to equations (20) and (21) in the form (10) and (11). The problem is naturally decomposed in the set of ordinary differential equations for the functions M_i, N_i, P_i and Q_i. Denoting by $\mathbf{B}_i$ the differential operator

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \left(4 - \frac{\mathrm{i}^2}{r^2}\right),$$

we can write this set as follows:

$$\mathbf{B}_i(M_i) = V_i^s, \quad \mathbf{B}_i(N_i) = W_i^s, \quad \mathbf{B}_i(P_i) = V_i^c, \quad \mathbf{B}_i(Q_i) = W_i^c, \tag{25}$$

and our aim is to give their explicit particular solutions in terms of $J_0(r)$ and $J_1(r)$.

Let us recall two identities concerning the Bessel function (see, for example, [8]).

$$\frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z), \tag{A}$$

or

$$J_{n+1}(z) = \frac{2n}{z} J_n(z) - J_{n-1}(z) \tag{A$'$}$$

$$J_n'(\dot{z}) = \tfrac{1}{2}(J_{n-1}(z) - J_{n+1}(z)). \tag{B}$$

Proposition 1. *For each pair of natural numbers* (i,k), $i \geqslant k \geqslant 0$, *the equation*

$$\mathbf{B}_i(Z) = J_k(r)J_{i-k}(r)$$

has a particular solution of the form

$$Q_{00}J_0^2 + Q_{01}J_0J_1 + Q_{11}J_1^2, \tag{26}$$

where Q_{AB} *are polynomials of* r^{-1} *and* r

$$Q_{AB} = \sum_{k=-n}^{m} C_k^{AB} r^k. \tag{27}$$

This solution will be denoted by $\mathbf{B}^+(i,k)$.

Proof. Let us consider two cases, i is even or odd:

Case $i = 2q$. Consider the $(q+1)$-dimensional vector space E_{2q} generated by $J_k(r)J_{2q-k}(r)$, $k = 0, 1, \ldots, q$. We have

$$\begin{aligned}\mathbf{B}_{2q}(J_k(r)J_{2q-k}(r)) &= 2\left(1 - \frac{(2q-k)k}{r^2}\right) J_{2q-k}(r)J_k(r) + 2J'_{2q-k}(r)J'_k(r)\\ &= -J_{k+1}(r)J_{2q-k-1}(r) + 2J_k(r)J_{2q-k}(r) - J_{k-1}(r)J_{2q-k+1}(r),\end{aligned}$$

where identities (A) and (B) are used. The functions $\mathbf{B}_{2q}(J_k(r)J_{2q-k}(r))$, $k = 1, \ldots, q$ generate the q-dimensional subspace E'_{2q} in E_{2q}. In addition, consider the function $K_{2q} = -\frac{r}{2}J_q(r)J'_q(\dot{r})$. It can also be checked that $\mathbf{B}_{2q}(K_{2q}) = J_q^2(r)$ (using again identities (A) and (B)) and that $J_q^2(r)$ does not belong to E'_{2q}. Therefore, the functions $\mathbf{B}_{2q}(K_{2q})$ and $\mathbf{B}_{2q}(J_k(r)J_{2q-k}(r))$, $k = 1, \ldots, q$, generate E_{2q}, which gives the required result after repeated use of identities (A′) and (B).

Case $i = 2q+1$. Consider the $(q+1)$-dimensional space vector E_{2q+1} generated by $J_k(r)J_{2q+1-k}(r)$, $k = 0, 1, \ldots, q$. Arguments similar to that in the previous case show that

$$\mathbf{B}_{2q+1}(J_k(r)J_{2q+1-k}(r)) = -J_{k+1}(r)J_{2q-k}(r) + 2J_k(r)J_{2q-k+1}(r) - J_{k-1}(r)J_{2q-k+2}(r).$$

The functions $\mathbf{B}_{2q+1}(J_k(r)J_{2q+1-k}(r))$, $k = 1, \ldots, q$ generate again the q-dimensional subspace E'_{2q} in E_{2q}. In addition, consider the function $K_{2q+1} = \left(\frac{q^2}{4r} - \frac{r}{4}\right)J_q^2(r) - \frac{q}{2}J_q(r)J'_q(\dot{r}) + \frac{r}{4}J'^2_q(\dot{r})$. It can also be checked that $\mathbf{B}_{2q+1}(K_{2q+1}) = J_q(r)J_{q+1}(r)$ and $J_q(r)J_{q+1}(r)$ does not belong to E'_{2q+1}. Therefore, the functions $\mathbf{B}_{2q+1}(K_{2q+1})$ and $\mathbf{B}_{2q+1}(J_k(r)J_{2q+1-k}(r))$, $k = 1, \ldots, q$ generate E_{2q+1}, which gives the required result after repeated use of identities (A′) and (B). □

Proposition 2. *For each pair of natural numbers* (i,k), $k \geqslant 0$, $i \geqslant 0$, *the equation*

$$\mathbf{B}_i(Z) = J_{i+k}(r)J_k(r)$$

has a particular solution of the form

$$Q_{00}J_0^2 + Q_{01}J_0J_1 + Q_{11}J_1^2, \tag{28}$$

where Q_{AB} *are polynomials of* r^{-1} *and* r

$$Q_{AB} = \sum_{k=-n}^{m} C_k^{AB} r^k. \tag{29}$$

This solution will be denoted by $\mathbf{B}^-(i,k)$.

Proof. It can easily be checked that

$$\mathbf{B}_i(J_{i+k}(r)J_k(r)) = J_{k-1}(r)J_{i+k-1}(r) + 2J_k(r)J_{i+k}(r) + J_{k+1}(r)J_{i+k+1}(r).$$

Therefore, $J_k(r)J_{i+k}(r)$ can be represented as a linear combination of the following functions: $\mathbf{B}_i(J_{i+l}(r)J_l(r)), l = 1, \ldots, k-1$, $J_{-1}(r)J_{i-1}(r)$ and $J_{-2}(r)J_{i-2}(r)$. But

$$J_{-1}(r)J_{i-1}(r) = -J_1(r)J_{i-1}(r) \qquad \text{and} \qquad J_{-2}(r)J_{i-2}(r) = J_2(r)J_{i-2}(r).$$

So, according to proposition 1, $J_{-1}(r)J_{i-1}(r)$ and $J_{-2}(r)J_{i-2}(r)$ can be represented as a linear combination of $\mathbf{B}_i(J_k(r)J_{i-k}(r))$, which proves proposition 2. □

Algorithm for constructing the nonlinear solutions

Step 1. The coefficients a_i, b_i, c_i and d_i defining the solutions $S_1(r,\theta)$ and $C_1(r,\theta)$ of the classical linear problem (see (8) and (9)) are determined using boundary conditions.

Step 2. The coefficients $\alpha_{ij}^{Vs}, \beta_{ij}^{Vs}, \alpha_{ij}^{Ws}, \beta_{ij}^{Ws}, \alpha_{ij}^{Vc}, \beta_{ij}^{Vc}, \alpha_{ij}^{Wc}, \beta_{ij}^{Wc}$ are determined substituting (8) and (9) into (23), collecting the similar terms, and using trigonometric transformations (see (24)). Thus the right-hand sides of equations (20) and (21) are calculated.

Step 3. According to propositions 1 and 2, the expressions

$$\begin{aligned} S_2^{\text{part}} &= \sum_{i=0}^{2N}\left[\left(\sum_{j=0}^{i}\alpha_{ij}^{Vs}\mathbf{B}^{+}(i,j) + \sum_{j=0}^{N-i}\beta_{ij}^{Vs}\mathbf{B}^{-}(i,j)\right)\sin \mathrm{i}\theta \right.\\ &\qquad \left. + \left(\sum_{j=0}^{i}\alpha_{ij}^{Ws}\mathbf{B}^{+}(i,j) + \sum_{j=0}^{N-i}\beta_{ij}^{Ws}\mathbf{B}^{-}(i,j)\right)\cos \mathrm{i}\theta\right] \\ C_2^{\text{part}} &= \sum_{i=0}^{2N}\left[\left(\sum_{j=0}^{i}\alpha_{ij}^{Wc}\mathbf{B}^{+}(i,j) + \sum_{j=0}^{N-i}\beta_{ij}^{Wc}\mathbf{B}^{-}(i,j)\right)\sin \mathrm{i}\theta \right.\\ &\qquad \left. + \left(\sum_{j=0}^{i}\alpha_{ij}^{Wc}\mathbf{B}^{+}(i,j) + \sum_{j=0}^{N-i}\beta_{ij}^{Wc}\mathbf{B}^{-}(i,j)\right)\cos \mathrm{i}\theta\right] \end{aligned} \tag{30}$$

give the solutions to equations (20) and (21).

Step 4. Values on the boundary of the derived nonlinear expressions are calculated.

Step 5. A pair of general solutions $S_2^{\text{hom}}(r,\theta)$ and $C_2^{\text{hom}}(r,\theta)$ to the homogenous equation,

$$Z_{2rr} + \frac{1}{r}Z_{2r} + \frac{1}{r^2}Z_{2\theta\theta} + 4Z_2 = 0, \tag{31}$$

is chosen in such a manner that the resulting solutions satisfy the boundary conditions.

Then

$$\begin{aligned} &\varepsilon S_1\sin(\omega t) + \varepsilon^2\left(\tfrac{1}{2}\left(S_1^2 - C_1^2\right) + S_2^{\text{part}} + S_2^{\text{hom}}\right)\sin(2\omega t) + \varepsilon C_1\cos(\omega t) \\ &\qquad + \varepsilon^2\left(S_1C_1 + C_2^{\text{part}} + C_2^{\text{hom}}\right)\cos(2\omega t) \end{aligned} \tag{32}$$

is a required solution to equation (2).

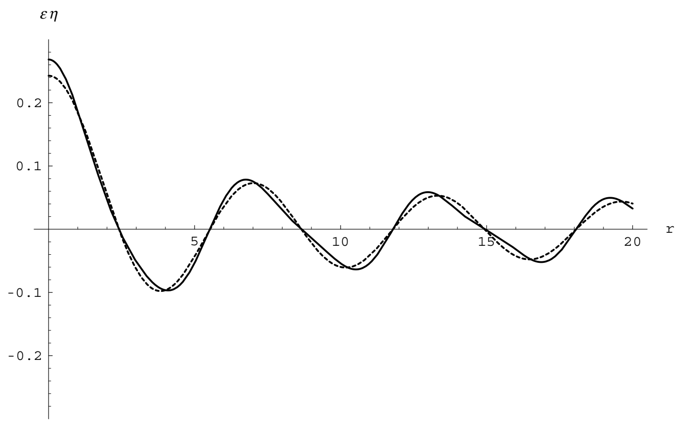

Figure 1. Axisymmetric standing waves. Dependence of surface elevation $\varepsilon\eta$ on radius r. The solid line is ε^2-order solution and the dashed line is the first-order solution. $\omega = 0.6$, $\varepsilon = 0.2$.

4. Examples

Here I apply the developed technique for constructing the explicit nonlinear corrections for classic linear solutions (3)–(5) of Lamb.

Axisymmetric standing waves. In the axisymmetric case, the calculation according to the presented algorithm gives the following expression:

$$\varphi(r, \theta, t) = \varepsilon J_0(kr)\sin(\omega t) + \varepsilon^2\left(\frac{\omega}{2}J_0^2 + \frac{3\omega}{4}rJ_0J_1\right)\sin(2\omega t). \tag{33}$$

A numerical example, which gives the surface wave level in dependence on radius r, is shown on figure 1.

Simplest unsymmetrical water waves. In this case the potential is given by the expression

$$\varphi_1(r, \theta, t) = \varepsilon J_1(r)\cos\theta\sin(\omega t) + \varepsilon^2 S_2(r, \theta)\sin(2\omega t), \tag{34}$$

where

$$S_2(r, \theta) = \left(\frac{3\omega^3}{8}J_0^2 - \frac{3\omega^3}{8}rJ_0J_1 - \frac{\omega}{4}J_1^2\right) + \left(-\frac{3\omega^3}{8}rJ_0J_1 + \frac{\omega}{8}J_1^2\right)\cos 2\theta. \tag{35}$$

A numerical calculation gives the contours of surface waves shown in figure 2.

Unsymmetrical water waves (semidiurnal tide). A potential is given by the expression

$$\varphi_1(r, \theta, t) = \varepsilon J_2(r)\cos 2\theta\sin(\omega t) + \varepsilon^2 S_2(r, \theta)\sin(2\omega t), \tag{36}$$

where

$$S_2(r, \theta) = \left(J_0^2 - \left(\frac{3r}{8} + \frac{1}{r}\right)J_0J_1 + \left(-\frac{3}{4} + \frac{1}{r^2}\right)J_1^2\right) + \left(-\frac{1}{2}J_0^2 + \left(-\frac{2}{r} + \frac{3r}{8}\right)J_0J_1 + \left(-\frac{2}{r^2} + \frac{3}{4}\right)J_1^2\right)\cos 4\theta. \tag{37}$$

The contours of surface waves are shown in figure 3.

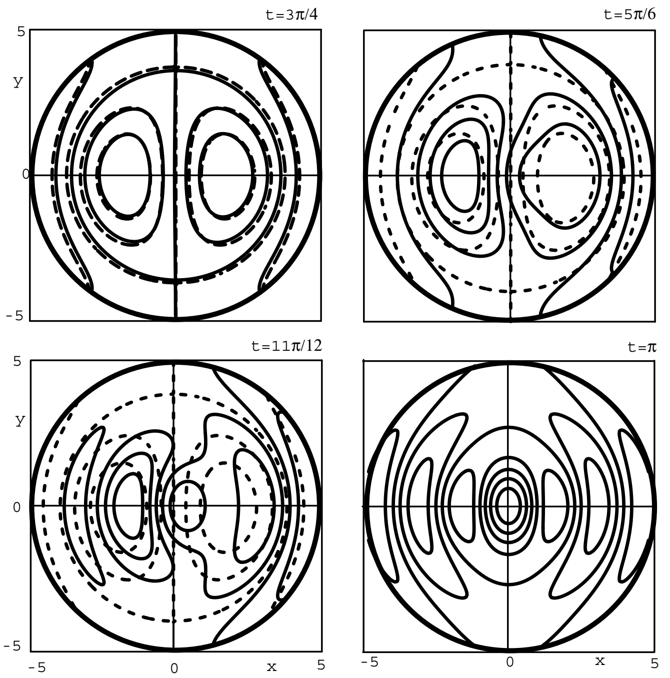

Figure 2. Contours of the simplest unsymmetric waves for $t = \frac{3}{4}\pi, \frac{5}{6}\pi, \frac{11}{12}\pi, \pi$. The solid lines are ε^2-order solutions and the dashed lines are ε-order solutions. $\omega = 1.0, \varepsilon = 0.2$.

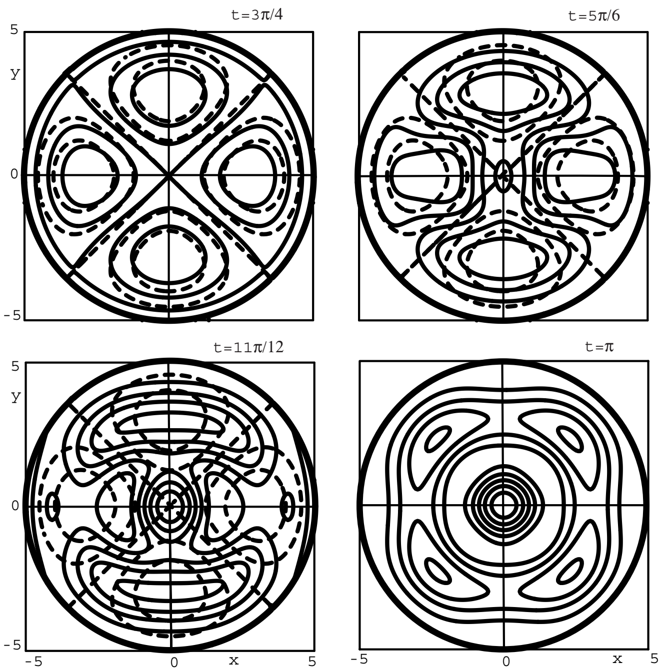

Figure 3. Contours of more complicated unsymmetric waves for $t = \frac{3}{4}\pi, \frac{5}{6}\pi, \frac{11}{12}\pi, \pi$. The solid lines are ε^2-order solutions and the dashed lines are ε-order solutions. $\omega = 1.0, \varepsilon = 0.2$.

5. Conclusions

An algorithm for constructing a family of nonlinear solutions to the nonlinear wave equation in polar coordinates has been presented.

The linear versions of these problems, when the terms only of first order in ε are retained, have been studied in the classic books (see [1, 2]).

The derived formulae are obtained by the method of unknown coefficients as solutions of some *overdetermined* systems of algebraic linear equations. The reason for their solvability remains obscure at the moment. Probably this is some kind of hidden symmetry.

A similar approach was used in [5, 6] for describing the long periodic water waves on a slope in the high-order shallow water approximation and in [7] for describing the first harmonic in θ for an isentropic gas flow in the non-dissipative case.

The linear generalized versions of equation (1) for scalar or vector potential are used in almost all branches of physics: from the elasticity theory to the plasma dynamics. Taking account for nonlinearity usually generates quadratic terms in the generalized equation (1). The author conjectures that the similar approach can be applied for studying a wide class of nonlinear wave equations in polar or cylindrical coordinates.

Acknowledgment

The author thanks the anonymous referees for helpful comments.

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